Partial Derivatives in Economics

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- Just as derivatives describe "marginal" cost for single variable cost functions, partial derivatives of production functions can describe marginal product of different inputs.
- For example, the **output quantity** Q may depend on labor L and capital K which is denoted as $Q = F(L,K)$. The **marginal product of labor (MPL)** is the change in output due to an additional unit of labor which is $\Delta Q = F(L^*+1, K^*) - F(L^*, K^*)$.
- If the input is divisible, we can let it be as small as we want, then the marginal product of labor is defined as the limit of the ratio of change in output and change in labor which is

$$\lim_{\Delta L \to 0} \frac{\Delta Q}{\Delta L} = \frac{F(L^*+\Delta L, K^*) - F(L^*, K^*)}{\Delta L} = \frac{\partial F}{\partial L}(L^*, K^*).$$

- In the same way, we define the **marginal product of capital (MPK)** as the partial derivative of the production function with respect to K.

1. Consider the Cobb-Douglas production function $Q = F(L,K) = 3L^{\frac{1}{3}}K^{\frac{2}{3}}$.

   (a) What is the output Q when $L = 125$ and $K = 1000$? Compute the marginal product of labor and marginal product of capital at $(L,K) = (125, 1000)$.

   $F(125,1000) = 3(125)^{\frac{1}{3}}(1000)^{2/3} = 1500.$

   $\frac{\partial F}{\partial L} = L^{-\frac{2}{3}}K^{\frac{2}{3}}$. $\frac{\partial F}{\partial L}(125,1000) = \frac{100}{25} = 4.$ $\frac{\partial F}{\partial K} = 2 \cdot L^{\frac{1}{3}}K^{-\frac{1}{3}}$. $\frac{\partial F}{\partial K}(125,1000) = \frac{2 \cdot 5}{10} = 1$

   (b) Use linear approximation to estimate the production Q when $L = 128$ and $K = 998$. Then compute $F(128, 998)$ with a calculator.

   The linear approximation of F at (125,1000) is

   $F(125,1000) + F_L(125,1000)(L-125) + F_K(125,1000)(K-1000)$

   $= 1500 + 4(L-125) + (K-1000)$. Hence $F(128,998) \approx 1500 + 4 \times 3 + (-2) = 1510$

   Wolframalpha gives the answer $F(128,998) = 1509.888714\ldots$

2. The average product of labor is $\frac{F(L,K)}{L}$, and the average product of capital is $\frac{F(L,K)}{K}$.

   (a) Find all production functions such that MPL is proportion to the average product of labor i.e. $\frac{\partial F}{\partial L} = \alpha_L \frac{F(L,K)}{L}$ for some constant $\alpha_L > 0$.

   If $\frac{\partial F}{\partial L} = \alpha_L \frac{F}{L}$, then $\frac{F_L}{F} = \alpha_L \frac{1}{L}$ i.e. $\frac{\partial}{\partial L}(\ln F) = \alpha_L \frac{\partial}{\partial L}(\ln(L))$.

   $\Rightarrow \frac{\partial}{\partial L}(\ln F - \ln L^{\alpha_L}) = 0$ which means that $\ln \frac{F(L,K)}{L^{\alpha_L}}$ depends on variable K only. Thus $\frac{F(L,K)}{L^{\alpha_L}} = g(K)$ for some function of K and $F(L,K) = L^{\alpha_L} \cdot g(K)$.

   (b) For constants $\alpha_L, \alpha_K > 0$, consider production functions satisfying the properties:

i. Zero output when one lacks any of the inputs: $F = 0$ if $L = 0$ or $K = 0$.

ii. MPL is proportion to the average product of labor: $\dfrac{\partial F}{\partial L} = \alpha_L \dfrac{F(L,K)}{L}$.

iii. MPK is proportion to the average product of capital: $\dfrac{\partial F}{\partial K} = \alpha_K \dfrac{F(L,K)}{K}$.

Show that the Cobb-Douglas production function $F(L,K) = AL^{\alpha_L}K^{\alpha_K}$, $A =$ constant, is the **only** function that satisfies these properties.
If we further require $F(\lambda L, \lambda K) = \lambda F(L,K)$ for all $\lambda > 0$ (in this case, F is said to have *constant returns to scale*), what is $\alpha_L + \alpha_K$?

From #2(a), we know that (ii) implies that $F(L,K) = L^{\alpha_L} \cdot g(K)$, where $g(K)$ is some function of K only. Hence, $\frac{\partial F}{\partial K} = L^{\alpha_L} \cdot g'(K)$, $\frac{F}{K} = L^{\alpha_L} \cdot \frac{g(K)}{K}$.
Property (iii) implies that $L^{\alpha_L} \cdot g'(K) = \alpha_K \cdot L^{\alpha_L} \cdot \frac{g(K)}{K} \Rightarrow g'(K) = \alpha_K \cdot \frac{g(K)}{K}$.
It is a diff equation for $g(K)$ and the solution is $g(K) = A \cdot K^{\alpha_K}$ for some constant A. Therefore (ii) and (iii) implies that $F(L,K) = L^{\alpha_L} \cdot g(K) = A \cdot L^{\alpha_L} \cdot K^{\alpha_K}$ which is a Cobb-Douglas production function.

If we further assume that $F(\lambda L, \lambda K) = \lambda F(L,K)$, then $F(\lambda L, \lambda K) = A\lambda^{\alpha_L + \alpha_K} L^{\alpha_L} \cdot K^{\alpha_K}$
$= \lambda \cdot A \cdot L^{\alpha_L} \cdot K^{\alpha_K}$ i.e. $\lambda^{\alpha_L + \alpha_K} = \lambda \Rightarrow \alpha_L + \alpha_K = 1$.

For a production function, the **Marginal Rate of Substitution (MRS)** of its inputs is the ratio of marginal products. Specifically, for production function $F(L,K)$ the MRS between labor L and capital K is $MRS_{LK} = \dfrac{MPL}{MPK}$ where $MPL = \dfrac{\partial F}{\partial L}$ and $MPK = \dfrac{\partial F}{\partial K}$.

1. Consider the level curve (indifference curve) $F(L,K) = Q_0$. Compute the slope of its tangent line, $\dfrac{dK}{dL}$, in terms of MRS_{LK}. What is the economic meaning of this slope?

By the implicit differentiation, $\frac{dK}{dL} = -\frac{F_L}{F_K}$ which by definition is $-\frac{MPL}{MPK} = -MRS_{LK}$. To maintain the same production ($F(L,K) = Q_0$), the reduction of one unit of labor must be replaced by increasing $\frac{F_L}{F_K}$ amount of capital.

2. Given $F(L,K) = L^{\frac{1}{3}}K^{\frac{2}{3}}$, compute MRS_{LK} at $(L,K) = (125, 1000)$. Find the tangent line of level curve $F(L,K) = 500$ at $(L,K) = (125, 1000)$ and determine concavity of the level curve.

$MRS_{LK} = \frac{MPL}{MPK} = \frac{\frac{1}{3}L^{-\frac{2}{3}}K^{\frac{2}{3}}}{\frac{2}{3}L^{\frac{1}{3}}K^{-\frac{1}{3}}} = \frac{1}{2}\frac{K}{L}$. At $(L,K) = (125, 1000)$,

$MRS_{LK} = \frac{1}{2}\frac{1000}{125} = 4$. The tangent line of $F(L,K) = 500$ at $(L,K) = (125, 1000)$ has slope $\frac{dK}{dL} = -MRS_{LK} = -4$. Hence the tangent line is $K - 1000 = -4(L - 125) \Rightarrow K + 4L = 1500$.

The level curve $F(L,K) = 500$ is $L^{\frac{1}{3}}K^{\frac{2}{3}} = 500 \Rightarrow K = (500)^{\frac{3}{2}}L^{-\frac{1}{2}}$.

$\frac{dK}{dL} = -\frac{1}{2}(500)^{\frac{3}{2}}L^{-\frac{3}{2}}$, $\frac{d^2K}{dL^2} = \frac{3}{4}(500)^{\frac{3}{2}}L^{-\frac{5}{2}} > 0$. Hence the level curve is concave upward.

Method of Lagrange Multipliers in Economics

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Consider the production function $q = f(L, K)$, where L is the amount of labor used and K is the amount of capital invested. Given that the unit cost of labor is P_L and the unit cost of capital is P_K, the total cost of production is $g(L, K) = P_L \cdot L + P_K \cdot K$. A firm may want to maximize production under fixed budget or minimize cost under fixed production.
To solve these optimization problems, we need the method of Lagrange multipliers.

1. Show that when the maximum production under fixed budget (or the minimum cost under fixed production) is obtained at (L^*, M^*), we have $\dfrac{\frac{\partial f}{\partial L}(L^*, K^*)}{P_L} = \dfrac{\frac{\partial f}{\partial K}(L^*, K^*)}{P_K}$. What is the economic interpretation of this equation?

By the method of Lagrange multipliers, if the maximum production under fixed budget is obtained at (L^*, K^*), $\vec{\nabla} f(L^*, K^*) = \lambda \vec{\nabla} g(L^*, K^*)$, which means that $f_L(L^*, K^*) = \lambda \cdot P_L$, $f_K(L^*, K^*) = \lambda \cdot P_K$. Thus $\dfrac{f_L(L^*, K^*)}{P_L} = \dfrac{f_K(L^*, K^*)}{P_K} = \lambda$.
This equation shows that when the extreme value is obtained, the last dollar used in labor and capital gives the same marginal product.

2. Consider the Cobb-Douglas production function $q = f(L, K) = AL^{\alpha}K^{1-\alpha}$, where $A > 0$, $0 < \alpha < 1$ are constants. Now the budget is fixed, $P_L \cdot L + P_K \cdot K = I$, where I is a constant.

   (a) Find the maximum production q_{max} in terms of I and α. Find the amount of labor and capital (L^*, K^*) which gives the maximum production and the Lagrange multiplier λ^* at the maximum point.

   (b) Compute $\dfrac{\partial q_{max}}{\partial I}$. Find the relation between $\dfrac{\partial q_{max}}{\partial I}$ and the Lagrange multiplier λ^*.

a) If the maximum production occurs at (L^*, K^*), $L^*, K^* > 0$,

then $\begin{cases} \vec{\nabla} f(L^*, K^*) = \lambda^* \vec{\nabla} g(L^*, K^*) \\ g(L^*, K^*) = I \end{cases} \Rightarrow \begin{cases} A\alpha L^{*\alpha-1} K^{*1-\alpha} = \lambda^* \cdot P_L & \text{— (1)} \\ A(1-\alpha) L^{*\alpha} K^{*-\alpha} = \lambda^* P_K & \text{— (2)} \\ P_L \cdot L^* + P_K \cdot K^* = I & \text{— (3)} \end{cases}$

$\dfrac{(1)}{(2)} \Rightarrow \dfrac{\alpha}{1-\alpha} \dfrac{K^*}{L^*} = \dfrac{P_L}{P_K} \Rightarrow K^* = \dfrac{1-\alpha}{\alpha} \dfrac{P_L}{P_K} \cdot L^*$ plug in (3)

$\Rightarrow P_L \cdot L^* + \dfrac{1-\alpha}{\alpha} P_L \cdot L^* = I \Rightarrow L^* = \alpha \cdot I / P_L$ and $K^* = (1-\alpha) I / P_K$,

$\lambda^* = A(1-\alpha) L^{*\alpha} K^{*-\alpha} \dfrac{1}{P_K} = A\left(\dfrac{\alpha}{P_L}\right)^{\alpha}\left(\dfrac{1-\alpha}{P_K}\right)^{1-\alpha}$. The maximum production is $q_{max}(I) = f(L^*, K^*) = A\left(\dfrac{\alpha I}{P_L}\right)^{\alpha}\left(\dfrac{(1-\alpha) I}{P_K}\right)^{1-\alpha}$

$= A \cdot I \cdot \left(\dfrac{\alpha}{P_L}\right)^{\alpha}\left(\dfrac{1-\alpha}{P_K}\right)^{1-\alpha}$. Hence $\dfrac{d q_{max}}{dI} = A\left(\dfrac{\alpha}{P_L}\right)^{\alpha}\left(\dfrac{1-\alpha}{P_K}\right)^{1-\alpha} = \lambda^*$.

3. Then suppose that the production is fixed, $\overline{q} = f(L, K) = AL^{\alpha}K^{1-\alpha}$, where $\overline{q}$ is a constant.

   (a) Find the minimum cost I_{min} in terms of $\overline{q}$ and α.

   (b) For fixed α, find the relation between $q_{max}(I)$ (from problem 2) and $I_{min}(\overline{q})$ (from this problem).

If the minimum cost occurs at (L^*, K^*), $L^*, K^* > 0$

then $\begin{cases} \vec{\nabla} g(L^*, K^*) = \lambda \vec{\nabla} f(L^*, K^*) \\ f(L^*, K^*) = \overline{q} \end{cases} \Rightarrow \begin{cases} P_L = \lambda A\alpha (L^*)^{\alpha-1}(K^*)^{1-\alpha} & — ① \\ P_K = \lambda A(1-\alpha) L^{*\alpha}(K^*)^{-\alpha} & — ② \\ A(L^*)^{\alpha}(K^*)^{1-\alpha} = \overline{q} & — ③ \end{cases}$

$\frac{①}{②} \Rightarrow \frac{P_L}{P_K} = \frac{\alpha}{1-\alpha}\frac{K^*}{L^*} \Rightarrow K^* = \frac{P_L}{P_K}\frac{1-\alpha}{\alpha}L^*$. Plug in ③ $\Rightarrow$

$A(L^*)^{\alpha}\cdot\left(\frac{P_L}{P_K}\frac{1-\alpha}{\alpha}\right)^{1-\alpha}\cdot(L^*)^{1-\alpha} = \overline{q} \Rightarrow L^* = \overline{q}\frac{1}{A}\cdot\left(\frac{P_K}{P_L}\frac{\alpha}{1-\alpha}\right)^{1-\alpha}$ and

$K^* = \overline{q}\frac{1}{A}\left(\frac{P_L}{P_K}\frac{1-\alpha}{\alpha}\right)^{\alpha}$. $I_{min}(\overline{q}) = P_L L^* + P_K\cdot K^* = \frac{\overline{q}}{A}P_L^{\alpha}P_K^{1-\alpha}\alpha^{-\alpha}(1-\alpha)^{\alpha-1}$.

Note that if $\overline{q} = q_{max}(I) = AI\left(\frac{\alpha}{P_L}\right)^{\alpha}\left(\frac{1-\alpha}{P_K}\right)^{1-\alpha}$, then $I_{min}(\overline{q}) = I$ i.e.

$I_{min}(q_{max}(I)) = I$.

4. Consider the production function $q = f(L, K) = L^{\frac{1}{2}} + K^{\frac{1}{2}}$. Find (L^*, K^*) that maximizes q subject to the budget constraint $P_L L + P_K K = I$, where $P_L, P_K > 0$ are constants.

   (a) If $P_L < P_K$, will $L^* > K^*$ or $L^* < K^*$?

   (b) Show that $\frac{L^*}{K^*}$ remains the same no matter how the budget I varies, i.e. the preference between labor and capital is invariant.

   (c) Consider a more general production function $f(L, K) = L^{\alpha} + K^{\alpha}$, where $\alpha > 0$ is a constant. Will $\frac{L^*}{K^*}$ remain the same no matter how I varies ?

(a) let $g(L, M) = P_L L + P_K\cdot K$. (L^*, K^*) satisfies $\begin{cases} \vec{\nabla} f(L^*, K^*) = \lambda\vec{\nabla} g \\ g(L^*, K^*) = I \end{cases}$

$\Rightarrow \begin{cases} \frac{1}{2}(L^*)^{-\frac{1}{2}} = \lambda P_L & — ① \\ \frac{1}{2}(K^*)^{-\frac{1}{2}} = \lambda P_K & — ② \\ P_L(L^*) + P_K(K^*) = I & — ③ \end{cases}$ $\quad \frac{①}{②} \Rightarrow \left(\frac{K^*}{L^*}\right)^{\frac{1}{2}} = \frac{P_L}{P_K}$.

$\Rightarrow \frac{K^*}{L^*} = \left(\frac{P_L}{P_K}\right)^2$.

If $P_L < P_K$, $\frac{K^*}{L^*} = \left(\frac{P_L}{P_K}\right)^2 < 1 \Rightarrow K^* < L^*$.

(b) For any constant I, $\frac{K^*}{L^*} = \left(\frac{P_L}{P_K}\right)^2$ is fixed.

(c) Suppose that $f(L, K) = L^{\alpha} + K^{\alpha}$ for some $\alpha > 0$. Then (L^*, K^*) satisfies $\begin{cases} \alpha(L^*)^{\alpha-1} = \lambda P_L \\ \alpha(K^*)^{\alpha-1} = \lambda P_K \end{cases} \Rightarrow \left(\frac{L^*}{K^*}\right)^{\alpha-1} = \frac{P_L}{P_K}$.

$\Rightarrow \frac{L^*}{K^*} = \left(\frac{P_L}{P_K}\right)^{\frac{1}{\alpha-1}}$ is fixed for any I.

Applications in Probability II

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- Consider random variables $X_i, 1 \leq i \leq n$ that take values on real numbers. Suppose that $f(x_1, x_2, \ldots, x_n)$ is an nonnegative function defined on R^n such that
  - $\int \cdots \int_{R^n} f(x_1, \ldots, x_n)\, dx_1 \cdots dx_n = 1.$
  - For any subset $V \subset R^n$, the probability of $(X_1, \ldots, X_n) \in V$ is

$$P((X_1, \ldots, X_n) \in V) = \int \cdots \int_V f(x_1, \ldots, x_n)\, dx_1 \cdots dx_n.$$

  Then $f(x_1, \ldots, x_n)$ is called the *joint probability density function* (joint p.d.f.) of $X_1, \ldots, X_n$.

- Suppose that each random variable X_i has probability density function $f_i(x_i)$. If the joint p.d.f. $f(x_1, \ldots, x_n) = f_1(x_1) \cdot f_2(x_2) \cdots f_n(x_n)$, then we say that the random variables $X_1, \ldots, X_n$ are **mutually independent**.

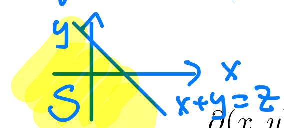
- For mutually independent random variables $X_1, \ldots, X_n$, we are interested in the new random variable $Z = X_1 + X_2 + \cdots + X_n$. For example, we would like to derive the p.d.f. of Z. Hence we need to find its distribution function, $F(z) = P(Z \leq z)$, and techniques of multiple integrals and change of variables are required.

1. Suppose that random variables X, Y are independent with p.d.f. $f_X(x), f_Y(y)$ respectively. Let $Z = X + Y$.

   (a) Represent $P(Z \leq z)$ as a double integral, $\iint_S f_X(x) f_Y(y) dA$. Draw the region S.

   $P(Z \leq z) = P(X+Y \leq z) = \iint_S f_X(x) f_Y(y)\, dA$, where

   S is the region $\{(x,y) \mid x+y \leq z\}$.

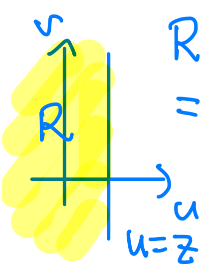


   (b) Try the change of variables $u = x+y, v = x$. Compute the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$. Rewrite $P(Z \leq z)$ as $\int_a^b \int_c^d g(u,v)\, dv du$.

   $\begin{cases} x = v \\ y = u - v \end{cases}$. $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = -1$.

   S corresponds to the region R in the uv-plane, where $R = \{(u,v) \mid u \leq z\}$. Hence $P(Z \leq z) = \iint_S f_X(x) f_Y(y)\, dA$

   $= \iint_R f_X(v) f_Y(u-v) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dA = \int_{-\infty}^{z} \int_{-\infty}^{\infty} f_X(v) f_Y(u-v)\, dv\, du$



   (c) Derive the p.d.f. of Z. Show that $\frac{d}{dz} P(Z \leq z) = \int_{-\infty}^{\infty} f_X(v) f_Y(z-v)\, dv$

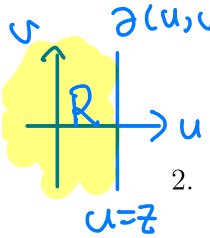
   The p.d.f. of Z is $\frac{d}{dz} P(Z \leq z) = \frac{d}{dz} \int_{-\infty}^{z} \int_{-\infty}^{\infty} f_X(v) f_Y(u-v)\, dv du$

   $= \int_{-\infty}^{\infty} f_X(v) f_Y(z-v)\, dv$

   (By the Fundamental Theorem of Calculus)

(d) Show that $\frac{d}{dz}P(Z \le z) = \int_{-\infty}^{\infty} f_X(z-v)f_Y(v)\, dv$. (Hint: Try the substitution $u = x+y, v = y$.) The substitution $u=x+y$, $v=y \iff x=u-v$, $y=v$.

$\frac{\partial(x,y)}{\partial(u,v)} = \left| \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} \right| = 1$. S corresponds to the region R in the uv-plane.



$$\frac{d}{dz}P(Z \le z) = \frac{d}{dz}\iint_S f_X(x)f_Y(y)\, dA = \frac{d}{dz}\iint_R f_X(u-v)f_Y(v)\, dA$$
$$= \int_{-\infty}^{\infty} f_X(z-v)f_Y(v)\, dv du$$

2. Examples with normal distributions:

(a) Suppose that $f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_1}e^{-\frac{x^2}{2\sigma_1^2}}$, and $f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma_2}e^{-\frac{y^2}{2\sigma_2^2}}$. Find the p.d.f. of $Z = X+Y$.

By Prob 1, we know that the p.d.f. of $Z=X+Y$ is

$$\int_{-\infty}^{\infty} f_X(v)f_Y(z-v)\, dv = \frac{1}{2\pi\sigma_1\sigma_2}\int_{-\infty}^{\infty} e^{-\frac{v^2}{2\sigma_1^2}-\frac{(z-v)^2}{2\sigma_2^2}}\, dv$$

The further computation is on the next page.

(b) Suppose that $f_{X_i}(x_i) = \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{x_i^2}{2\sigma^2}}$, for $1 \le i \le n$, and $X_1, \ldots, X_n$ are mutually independent. Find the p.d.f. of $Z = X_1 + X_2 + \cdots + X_n$.

The solution is on the next page.

3. Suppose that for $1 \le i \le n$, $f_{X_i}(x_i) = e^{-x_i}$ if $x_i \ge 0$ and $f_{X_i}(x_i) = 0$ if $x_i < 0$, and $X_1, \ldots X_n$ are mutually independent. Find the p.d.f. of $Z_2 = X_1 + X_2$ and $Z_n = X_1 + \cdots + X_n$.

The p.d.f. of Z_2 is $f_{Z_2}(z) = \int_{-\infty}^{\infty} f_{X_1}(v)f_{X_2}(z-v)\, dv$

$\because f_{X_1}(v) = 0 = f_{X_2}(z-v)$ for $v<0$ or $v>z$.

$\therefore$ If $z<0$, $f_{Z_2}(z)=0$. If $z \ge 0$, $f_{Z_2}(z) = \int_0^z f_{X_1}(v)f_{X_2}(z-v)\, dv$

$= \int_0^z e^{-v}e^{-z+v}\, dv = z \cdot e^{-z}$.

The p.d.f. of $Z_3 = X_1+X_2+X_3 = Z_2 + X_3$ is

$f_{Z_3}(z) = \int_{-\infty}^{\infty} f_{Z_2}(v)f_{X_3}(z-v)\, dv = \int^z v \cdot e^{-v} \cdot e^{-z+v}\, dv = \frac{z^2}{2}e^{-z}$,

for $z \ge 0$ and $f_{Z_3}(z) = 0$, for $z<0$. By mathematical induction, the p.d.f. of $Z_n = X_1 + \cdots + X_n$ is $f_{Z_n}(z) = \frac{z^{n-1}}{(n-1)!}e^{-z}$ for $z \ge 0$ and $f_{Z_n}(z) = 0$ for $z<0$.

#2(a) : The p.d.f. of $Z = X+Y$ is

$$\frac{1}{2\pi\sigma_1\sigma_2}\int_{-\infty}^{\infty} e^{-\frac{v^2}{2\sigma_1^2}-\frac{(z-v)^2}{2\sigma_2^2}}\,dv$$

(complete the square.)

$$= \frac{1}{2\pi\sigma_1\sigma_2}\int_{-\infty}^{\infty} e^{-\frac{(\sigma_1^2+\sigma_2^2)}{2\sigma_1^2\sigma_2^2}\left(v-\frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}z\right)^2-\frac{z^2}{2(\sigma_1^2+\sigma_2^2)}}\,dv$$

$$= \frac{1}{2\pi\sigma_1\sigma_2}\, e^{-\frac{z^2}{2(\sigma_1^2+\sigma_2^2)}}\int_{-\infty}^{\infty} e^{-\frac{(\sigma_1^2+\sigma_2^2)}{2\sigma_1^2\sigma_2^2}\left(v-\frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}z\right)^2}\,dv$$

$$= \frac{1}{2\pi\sigma_1\sigma_2}\, e^{-\frac{z^2}{2(\sigma_1^2+\sigma_2^2)}}\int_{-\infty}^{\infty} e^{-u^2}\cdot\frac{\sqrt{2}\sigma_1\sigma_2}{\sqrt{\sigma_1^2+\sigma_2^2}}\,du$$

$$\left(\text{Let } u = \frac{\sqrt{\sigma_1^2+\sigma_2^2}}{\sqrt{2}\sigma_1\sigma_2}\left(v-\frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}z\right)\quad du = \frac{\sqrt{\sigma_1^2+\sigma_2^2}}{\sqrt{2}\sigma_1\sigma_2}\,dv\right)$$

$$= \frac{1}{\sqrt{2}\sqrt{\sigma_1^2+\sigma_2^2}\,\pi}\, e^{-\frac{z^2}{2(\sigma_1^2+\sigma_2^2)}}\int_{-\infty}^{\infty} e^{-u^2}\,du$$

$$= \frac{1}{\sqrt{2\pi}\,\sqrt{\sigma_1^2+\sigma_2^2}}\, e^{-\frac{z^2}{2(\sigma_1^2+\sigma_2^2)}} \qquad \left(\because \int_{-\infty}^{\infty} e^{-u^2}\,du = \sqrt{\pi}\right)$$

This means that $Z = X_1 + X_2$ is still a normally distributed variable with expected value $\mu = 0$ and standard deviation $\sigma = \sqrt{\sigma_1^2+\sigma_2^2}$.

#2 (b).

By #2(a), $Z_2 = X_1 + X_2$ is a normally distributed variable with $\mu = 0$ and standard deviation $\sqrt{\sigma^2 + \sigma^2} = \sqrt{2}\sigma$.

By #2(a), $Z_3 = X_1 + X_2 + X_3 = Z_2 + X_3$, is a normally distributed variable with $\mu = 0$ and standard deviation

$\sqrt{(\sqrt{2}\sigma)^2 + \sigma^2} = \sqrt{3}\sigma$.

Suppose that $Z_n = X_1 + \dots + X_n$ is a normally distributed variable with $\mu = 0$ and standard deviation $\sqrt{n}\sigma$.
Then $Z_{n+1} = X_1 + \dots + X_{n+1} = Z_n + X_{n+1}$ is a normally distributed variable with $\mu = 0$ and standard deviation

$\sqrt{(\sqrt{n}\sigma)^2 + \sigma^2} = \sqrt{n+1}\,\sigma$.

Hence by mathematical induction, we show that

Z_n has p.d.f. $f_{Z_n}(z) = \dfrac{1}{\sqrt{2n\pi}\,\sigma} e^{-\frac{z^2}{2n\sigma^2}}$ for all $n \in \mathbb{N}$.